FOL
FOLDNF und Elementare Quantorenlogik

VL: Finitismus

PD Dr. Timm Lampert

Humboldt Universität Berlin

FOL und Fragmente

- FOL: Junktoren $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ und Quantoren $\exists, \forall$ sowie n-stellige atomare Ausdrücke ($n \geq 0$)
- keine freien Variablen, keine Namen, keine Identität, keine mathematische Funktionen
- Frage: Inwieweit ist FOL algorithmisierbar?

$\Rightarrow$ Entscheidungsproblem: Inwieweit kann entschieden werden, ob eine FOL-Formel kontradiktorisch ist?

1. FOL-Fragmente
   1. Elementare FOL: keine dyadischen Junktoren im Wirkungsbereich
   2. Monadische FOL: $n < 2$.
2. FOL

Pränexe Normalform (PNF)

- Jede FOL-Formel φ kann durch
  - Anwendung von Quantordefinitionen, DMG, Pfeilelimination in NNF (negative Normform) gebracht werden,
  - Anwendung von PN-Gesetzen in pränexe Normform gebracht werden.
- Während in der Aussagenlogik DNF und CNF dominieren, dominieren in FOL pränexe Normalformen die metalogische Diskussion.

PN-Definitionen

Definition of Negation. We have first to define the negations of $(x).\phi x$ and $(\exists x).\phi x$. We define the negation of $(x).\phi x$ as $(\exists x).\sim\phi x$, *i.e.* "it is not the case that ϕx is always true" is to mean "it is the case that not-ϕx is sometimes true." Similarly the negation of $(\exists x).\phi x$ is to be defined as $(x).\sim\phi x$. Thus we put

$*9\cdot01.\quad \sim\{(x).\phi x\}.=.(\exists x).\sim\phi x \quad \mathrm{Df}$

$*9\cdot02.\quad \sim\{(\exists x).\phi x\}.=.(x).\sim\phi x \quad \mathrm{Df}$

Definition of Disjunction. To define disjunction when one or both of the propositions concerned is of the first order, we have to distinguish six cases, as follows:

$*9\cdot03.\quad (x).\phi x.\vee.p:=.(x).\phi x\vee p \quad \mathrm{Df}$

$*9\cdot04.\quad p.\vee.(x).\phi x:=.(x).p\vee\phi x \quad \mathrm{Df}$

$*9\cdot05.\quad (\exists x).\phi x.\vee.p:=.(\exists x).\phi x\vee p \quad \mathrm{Df}$

$*9\cdot06.\quad p.\vee.(\exists x).\phi x:=.(\exists x).p\vee\phi x \quad \mathrm{Df}$

Priorität pränexer Normalform

Russell & Whitehead, *Principia Mathematica*, S. 136:

„[…] the true scope of an apparent variable is always the whole of the asserted proposition in which it occurs, even when, typographically, its scope appears to be only part of the asserted proposition. Thus when $(\exists x).\phi x$ or $(x).\phi x$ appears as part of an asserted proposition, it does not really occur, since the scope of the apparent variable really extends to the whole asserted proposition.“

PN-Gesetze

$\forall\nu(A \land B(\nu))$	$\dashv\vdash$	$A \land \forall\nu B(\nu)$	PN1
$\forall\nu(B(\nu) \land A)$	$\dashv\vdash$	$\forall\nu B(\nu) \land A$	PN2
$\forall\nu(A \lor B(\nu))$	$\dashv\vdash$	$A \lor \forall\nu B(\nu)$	PN3
$\forall\nu(B(\nu) \lor A)$	$\dashv\vdash$	$\forall\nu B(\nu) \lor A$	PN4
$\exists\nu(A \land B(\nu))$	$\dashv\vdash$	$A \land \exists\nu B(\nu)$	PN5
$\exists\nu(B(\nu) \land A)$	$\dashv\vdash$	$\exists\nu B(\nu) \land A$	PN6
$\exists\nu(A \lor B(\nu))$	$\dashv\vdash$	$A \lor \exists\nu B(\nu)$	PN7
$\exists\nu(B(\nu) \lor A)$	$\dashv\vdash$	$\exists\nu B(\nu) \lor A$	PN8
$\forall\nu(A(\nu) \land B(\nu))$	$\dashv\vdash$	$\forall\nu A(\nu) \land \forall\nu B(\nu)$	PN9
$\exists\nu(A(\nu) \lor B(\nu))$	$\dashv\vdash$	$\exists\nu A(\nu) \lor \exists\nu B(\nu)$	PN10

Klassifikationsproblem

- Unterscheidung entscheidbarer pränexer Normalformen und unentscheidbarer pränexer Normalformen.
- Unentscheidbare Klasse: Klasse unendlich vieler Formeln, in der nicht alle Formeln entscheidbar sind.
- Beweis:
  - \+ Angabe eines Entscheidungsverfahrens
  - – durch Reduktion auf Klassen, die Formeln enthalten, durch die unentscheidbare Probleme repräsentiert werden.

Unentscheidbare Fälle

Classes with finite prefix:

- $[\forall\exists\forall, (\omega, 1)]$ *(Kahr 1962)*
- $[\forall^3\exists, (\omega, 1)]$ *(Surányi 1959)*

Classes with $\forall^*$ *in the prefix:*

- $[\forall^*\exists, (0, 1)]$ *(Kalmár-Surányi 1950)*
- $[\forall\exists\forall^*, (0, 1)]$ *(Denton 1963)*

Classes with $\exists^*$ *in the prefix:*

- $[\forall\exists\forall\exists^*, (0, 1)]$ *(Gurevich 1966)*
- $[\forall^3\exists^*, (0, 1)]$ *(Kalmár-Surányi 1947)*
- $[\forall\exists^*\forall, (0, 1)]$ *(Kostyrko-Genenz 1964)*
- $[\exists^*\forall\exists\forall, (0, 1)]$ *(Surányi 1959)*
- $[\exists^*\forall^3\exists, (0, 1)]$ *(Surányi 1959)*

Reduktion

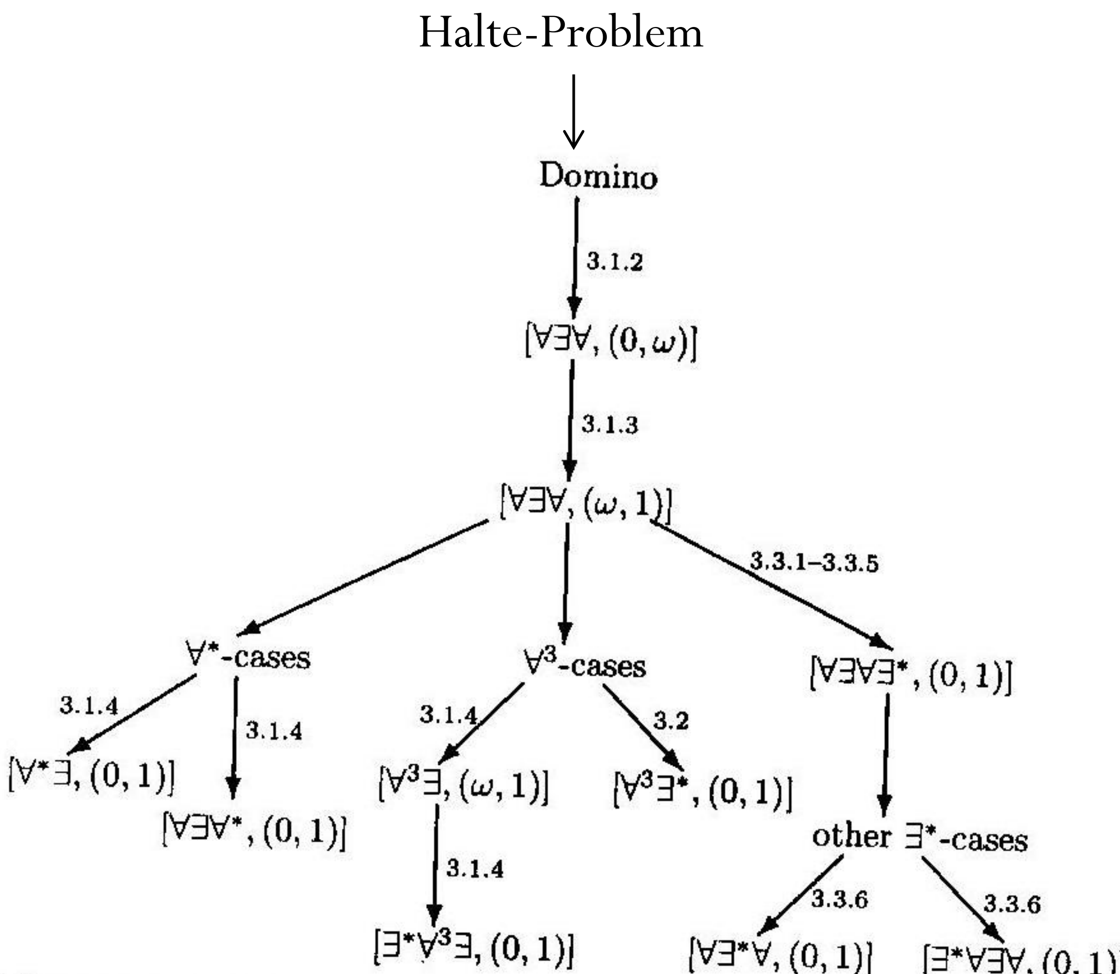


Figure 3.1. Reductions for standard classes

Entscheidbare Fälle

(1)	$[\exists^*\forall^*, all, (0)]_=$	(Ramsey 1930)
(2)	$[\exists^*\forall^2\exists^*, all, (0)]$	(Gödel 1932, Kalmár 1933, Schütte 1934)
(3)	$[all, (\omega), (\omega)]$	(Löb 1967, Gurevich 1969)
(4)	$[\exists^*\forall\exists^*, all, all]$	(Maslov-Orevkov 1972, Gurevich 1973)

Alle entscheidbar mittels endlicher Modelle.

Unentscheidbarkeit

- Die Annahme der Unentscheidbarkeit von FOL hängt ab von $\mathfrak{I}$(FOL) und dem Verständnis von Turing-berechenbare oder μ-rekursiven Funktionen als Funktionen im extensionalen Sinne.
- Diese Voraussetzungen werden im Folgenden nicht geteilt. Wir setzten die Kritik am Unentscheidbarkeitsbeweis voraus.
- Wir diskutieren auch nicht mehr das unterschiedliche Verständnis von FOL in der mathematischen Logik und im Finitismus.
- Wir fragen uns vielmehr, wie das algorithmische Beweisverständnis umgesetzt werden kann.

Anti-pränexe Normalform (FOLDNF)

FOLDNF: DNF, in denen die Quantoren durch PN-Gesetze maximal nach innen gezogen werden.

Vgl. Quine's purified normal forms in der 4. Auflage von *Methods of Logic*, S.126f.

Methode: Identifiziere logische Beziehungen anhand der syntaktischen (formalen) Eigenschaften von FOLDNF.

Analogie zur Aussagenlogik: Darstellung der Wahrheitsbedingungen anhand von DNF bzw. ab-Polgruppen.

Beispiel: *Elementare* Quantorenlogik – Verallgemeinerung des Quine-McCluskey Algorithmus.

Logische Beweise

φ → ab-Zeichen → ab-Polgruppen → ab-Minpolgruppen → ab-Primpolgruppen: ab-Symbol.

Quine McCluskey (1. Schritt)

φ → KDNF → MINDNF → RDNF.

Allgemeiner Algorithmus zur Lösung des Äquivalenzproblems.

Paraphrase

φ: $(P\wedge Q)\vee(P\wedge\neg Q)$ => a-{a-P}, b-{b-P}

Paraphrase: Eine Instanz von

$(P\wedge Q)\vee(P\wedge\neg Q)$ ist

- wahr gdw. P der Fall ist und
- falsch gdw. P nicht der Fall ist.

- Das Resultat eines logischem Beweises ist eine logische Form, die eineindeutig als Darstellung von Wahrheitsbedingungen interpretiert werden kann.

Quantordefinitionen

$$*9.01 \quad \sim (x).\phi x. = .(\exists x). \sim \phi x \quad Df$$

$$b - a - \forall x - a\phi xb - \exists x - b - a \qquad\qquad b - \forall x - b - a\phi xb - a - \exists x - a$$

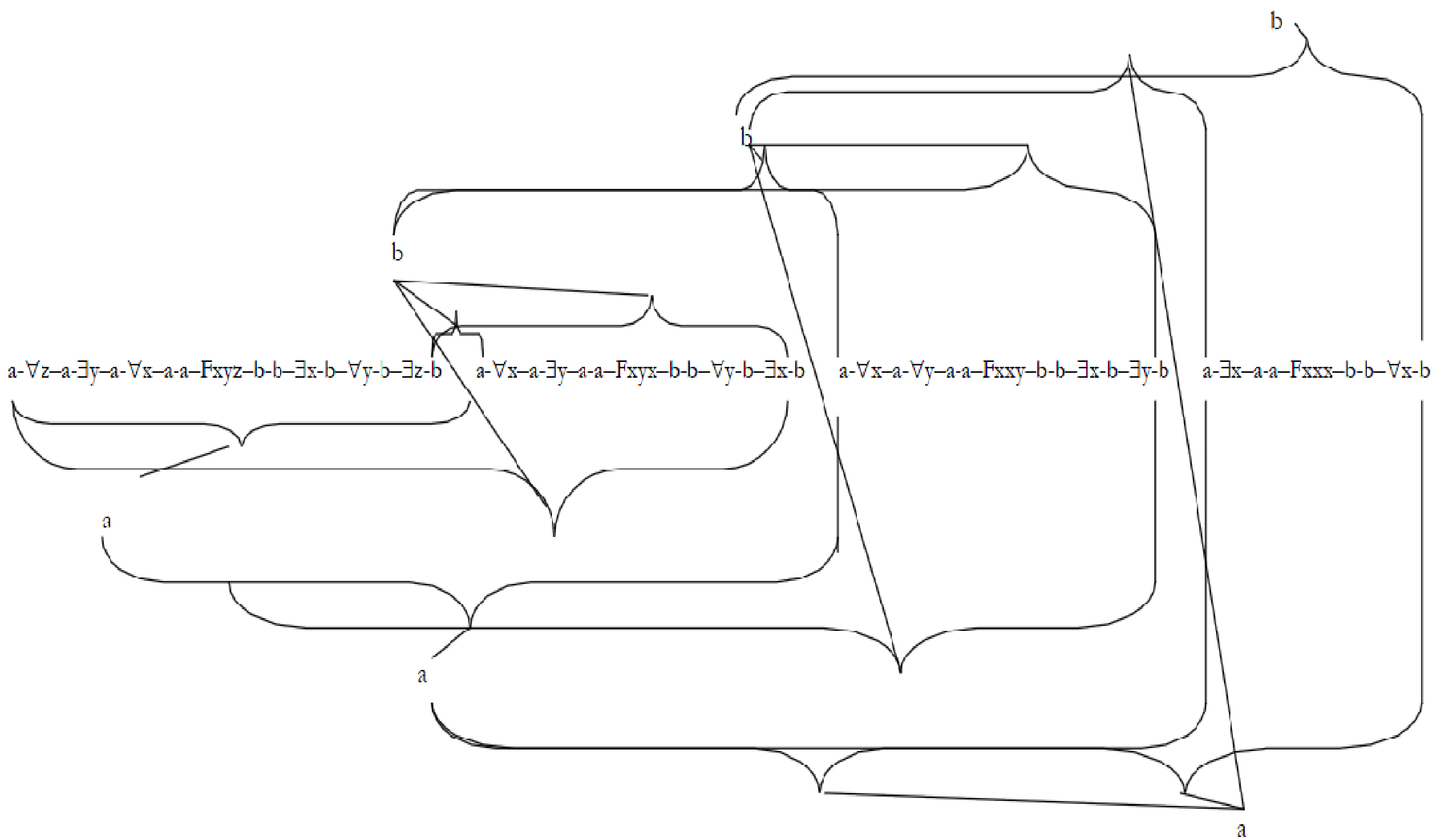
$$\begin{aligned} a &- \{\exists x - b - \phi x\}, \\ b &- \{\forall x - a - \phi x\}. \end{aligned}$$

$$\begin{aligned} a &- \{\exists_1 - b - \phi_1\}, \\ b &- \{\forall_1 - a - \phi_1\}. \end{aligned}$$

NL, S. 192: „Frühere Definitionen werden jetzt tautologisch."

ab-Zeichen

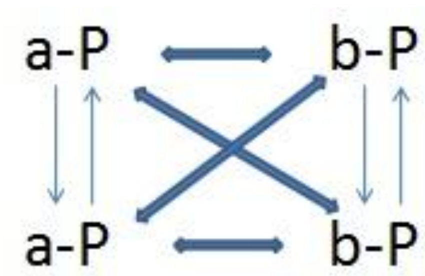
$$\forall z\exists y\forall x Fxyz \wedge \forall x\exists y Fxyx \wedge \forall x\forall y Fxxy \vee \exists x Fxxx$$

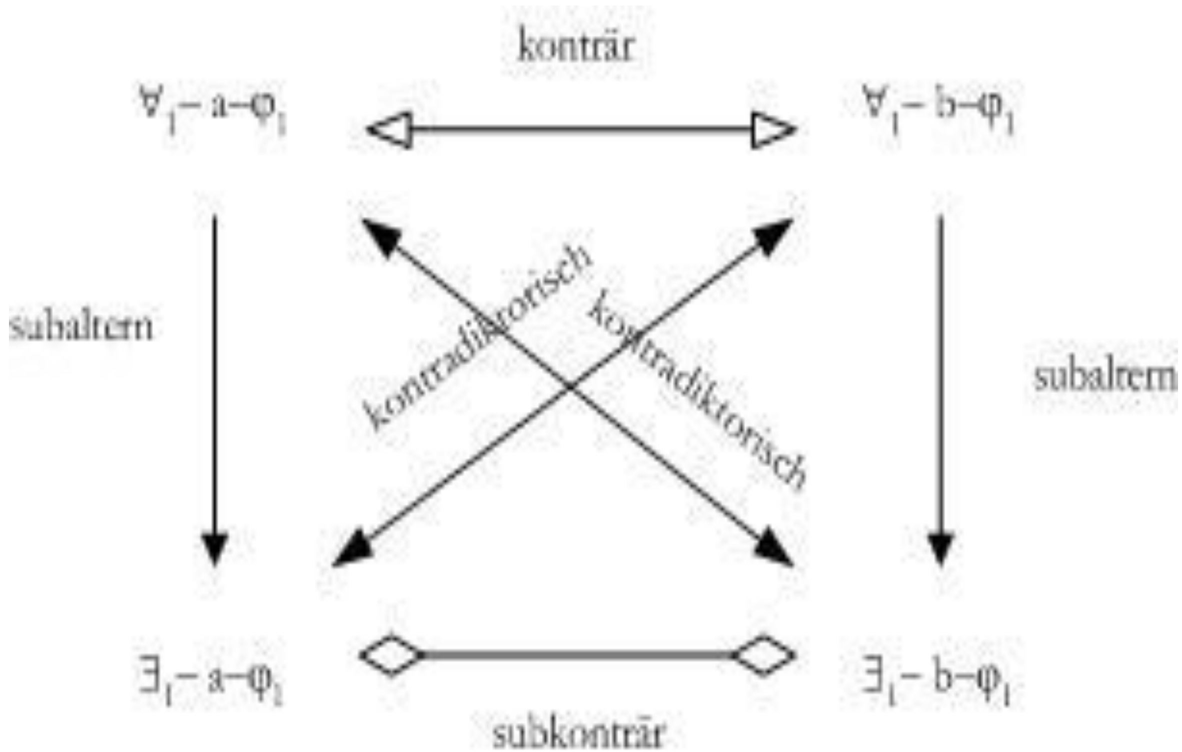


Polgruppen

1. $a - \{\forall_3\exists_2\forall_1 - a - F_{123}, \quad \forall <^{1}_{3}\exists_2 - a - F_{123}, \quad \forall <^{1}_{2},\forall_3 - a - F_{123}, \quad \exists <^{\frac{1}{2}}_{3} - a - F_{123}\},$
2. $a - \{\forall_3\exists_2\forall_1 - a - F_{123}, \quad \forall <^{1}_{3}\exists_2 - a - F_{123}, \quad \forall <^{1}_{2},\forall_3 - a - F_{123}, \quad \forall <^{\frac{1}{2}}_{3} - b - F_{123}\},$
3. $a - \{\forall_3\exists_2\forall_1 - a - F_{123}, \quad \forall <^{1}_{3}\exists_2 - a - F_{123}, \quad \exists <^{1}_{2},\exists_3 - b - F_{123}, \quad \exists <^{\frac{1}{2}}_{3} - a - F_{123}\},$
4. $a - \{\forall_3\exists_2\forall_1 - a - F_{123}, \quad \exists <^{1}_{3}\forall_2 - b - F_{123}, \quad \forall <^{1}_{2},\forall_3 - a - F_{123}, \quad \exists <^{\frac{1}{2}}_{3} - a - F_{123}\},$
5. $a - \{\exists_3\forall_2\exists_1 - b - F_{123}, \quad \forall <^{1}_{3}\exists_2 - a - F_{123}, \quad \forall <^{1}_{2},\forall_3 - a - F_{123}, \quad \exists <^{\frac{1}{2}}_{3} - a - F_{123}\},$
6. $b - \{\forall_3\exists_2\forall_1 - a - F_{123}, \quad \forall <^{1}_{3}\exists_2 - a - F_{123}, \quad \exists <^{1}_{2},\exists_3 - b - F_{123}, \quad \forall <^{\frac{1}{2}}_{3} - b - F_{123}\},$
7. $b - \{\forall_3\exists_2\forall_1 - a - F_{123}, \quad \exists <^{1}_{3}\forall_2 - b - F_{123}, \quad \forall <^{1}_{2},\forall_3 - a - F_{123}, \quad \forall <^{\frac{1}{2}}_{3} - b - F_{123}\},$
8. $a - \{\forall_3\exists_2\forall_1 - a - F_{123}, \quad \exists <^{1}_{3}\forall_2 - b - F_{123}, \quad \exists <^{1}_{2},\exists_3 - b - F_{123}, \quad \exists <^{\frac{1}{2}}_{3} - a - F_{123}\},$
9. $b - \{\exists_3\forall_2\exists_1 - b - F_{123}, \quad \forall <^{1}_{3}\exists_2 - a - F_{123}, \quad \forall <^{1}_{2},\forall_3 - a - F_{123}, \quad \forall <^{\frac{1}{2}}_{3} - b - F_{123}\},$
10. $a - \{\exists_3\forall_2\exists_1 - b - F_{123}, \quad \forall <^{1}_{3}\exists_2 - a - F_{123}, \quad \exists <^{1}_{2},\exists_3 - b - F_{123}, \quad \exists <^{\frac{1}{2}}_{3} - a - F_{123}\},$
11. $a - \{\exists_3\forall_2\exists_1 - b - F_{123}, \quad \exists <^{1}_{3}\forall_2 - b - F_{123}, \quad \forall <^{1}_{2},\forall_3 - a - F_{123}, \quad \exists <^{\frac{1}{2}}_{3} - a - F_{123}\},$
12. $b - \{\forall_3\exists_2\forall_1 - a - F_{123}, \quad \exists <^{1}_{3}\forall_2 - b - F_{123}, \quad \exists <^{1}_{2},\exists_3 - b - F_{123}, \quad \forall <^{\frac{1}{2}}_{3} - b - F_{123}\},$
13. $b - \{\exists_3\forall_2\exists_1 - b - F_{123}, \quad \forall <^{1}_{3}\exists_2 - a - F_{123}, \quad \exists <^{1}_{2},\exists_3 - b - F_{123}, \quad \forall <^{\frac{1}{2}}_{3} - b - F_{123}\},$
14. $b - \{\exists_3\forall_2\exists_1 - b - F_{123}, \quad \exists <^{1}_{3}\forall_2 - b - F_{123}, \quad \forall <^{1}_{2},\forall_3 - a - F_{123}, \quad \forall <^{\frac{1}{2}}_{3} - b - F_{123}\},$
15. $a - \{\exists_3\forall_2\exists_1 - b - F_{123}, \quad \exists <^{1}_{3}\forall_2 - b - F_{123}, \quad \exists <^{1}_{2},\exists_3 - b - F_{123}, \quad \exists <^{\frac{1}{2}}_{3} - a - F_{123}\},$
16. $b - \{\exists_3\forall_2\exists_1 - b - F_{123}, \quad \exists <^{1}_{3}\forall_2 - b - F_{123}, \quad \exists <^{1}_{2},\exists_3 - b - F_{123}, \quad \forall <^{\frac{1}{2}}_{3} - b - F_{123}\}.$

Polbeziehungen



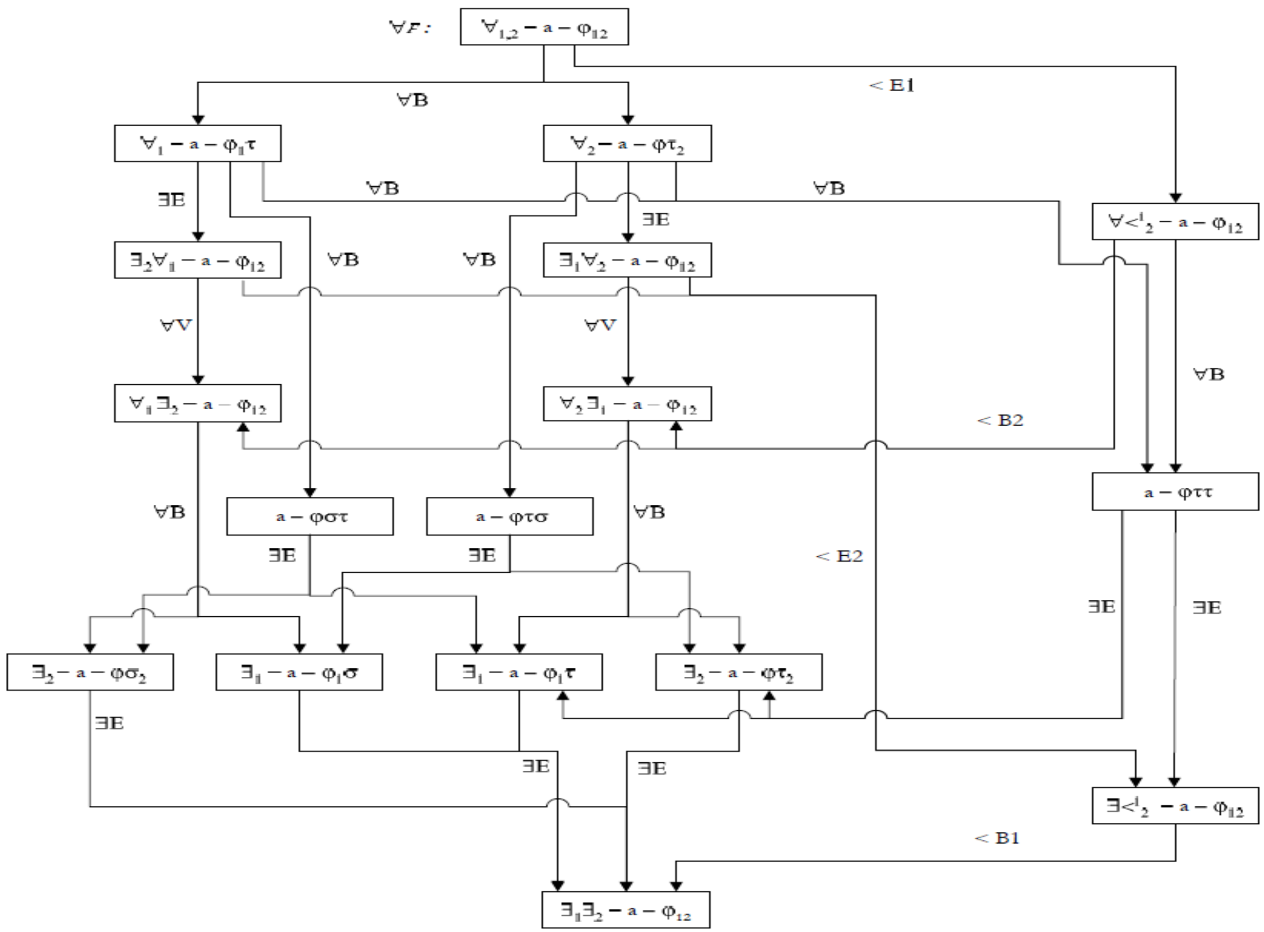


Kalkül: Elementare Pole

$\exists\forall Ex$:	$\exists\mu\forall\nu \vdash \forall\nu\exists\mu$		
$\forall E$:	$\forall\mu \vdash t\mu$	$\exists I$:	$t < {}^{\mu}_{\nu} \vdash t\mu, \exists\nu$
$< I1$:	$\forall\mu, \forall\nu \vdash \forall < {}^{\mu}_{\nu}$	$< E1$:	$\exists < {}^{\mu}_{\nu} \vdash \exists\mu, \exists\nu$
$< I2$:	$\exists\mu\forall\nu \vdash \exists < {}^{\mu}_{\nu}$	$< E2$:	$\forall < {}^{\mu}_{\nu} \vdash \forall\mu\exists\nu$

$\overline{\varphi/\psi}$:	$\phi \nvdash \psi$	$\overline{a/b}$:	$a - \varphi \nvdash b - \varphi$ $b - \varphi \nvdash a - \varphi$
$\overline{\forall\exists Ex}$:	$\forall\mu\exists\nu \nvdash \exists\nu\forall\mu$		
$\overline{tI}$:	$s\mu \nvdash t\mu$	$\overline{\exists E}$:	$\exists\mu \nvdash t\mu$
$\overline{< E1}$:	$\forall < {}^{\mu}_{\nu} \nvdash s\mu, t\nu$	$\overline{< I1}$:	$s\mu, t\nu \nvdash \exists < {}^{\mu}_{\nu}$
$\overline{< E2}$:	$\forall < {}^{\mu}_{\nu} \nvdash \exists\mu\forall\nu$	$\overline{< I2}$:	$\forall\mu\exists\nu \nvdash \exists < {}^{\mu}_{\nu}$

Implikationsbeziehungen



Klassifizierung

no.	variation	rule
1.	$\varphi \Rightarrow \psi$	$\overline{\varphi/\psi}$
2.	$a - \varphi \Rightarrow b - \varphi$ $b - \varphi \Rightarrow a - \varphi$	$\overline{a/bEx}$
3.	$\exists\mu\forall\nu \Rightarrow \forall\nu\exists\mu$	$\exists\forall Ex$
4.	$\forall\nu\exists\mu \Rightarrow \exists\mu\forall\nu$	$\overline{\forall\exists Ex}$
5.	$\forall\mu \Rightarrow t\mu$	$\forall E$
6.	$\forall\mu \Rightarrow \exists\mu$	$\forall E, \exists I$
7.	$s\mu \Rightarrow t\mu$	$\overline{tI}$
8.	$t\mu \Rightarrow \forall\mu$	a.f. 1.7: SC, $\forall E$
9.	$t\mu \Rightarrow \exists\mu$	$\exists I$
10.	$\exists\mu \Rightarrow t\mu$	$\overline{\exists E}$
11.	$\exists\mu \Rightarrow \forall\mu$	a.f. 1.10: SC, $\forall E$
12.	$\forall\mu, \forall\nu \Rightarrow \forall <^{\mu}_{\nu}$	$< I1$
13.	$* \Rightarrow \forall <^{\mu}_{\nu}$	1.8,11
14.	$\forall\mu, t\nu \Rightarrow t <^{\mu}_{\nu}$	$\forall E$
15.	$\forall\mu, \forall\nu \Rightarrow t <^{\mu}_{\nu}$	$< I1, \forall E$
16.	$* \Rightarrow t <^{\mu}_{\nu}$	1.7,10
17.	$\exists\mu\forall\nu \Rightarrow \exists <^{\mu}_{\nu}$	$< I2$
18.	$t\mu\forall\nu \Rightarrow \exists <^{\mu}_{\nu}$	$\exists I, \exists\forall Ex, < I2$
19.	$\forall\mu\forall\nu \Rightarrow \exists <^{\mu}_{\nu}$	$\forall E, \exists I, \exists\forall Ex, < I2$
20.	$\forall\mu\exists\nu \Rightarrow \exists <^{\mu}_{\nu}$	$\overline{< E2}$
21.	$s\mu, t\nu \Rightarrow \exists <^{\mu}_{\nu}$	$\overline{< I1}$

no.	variation	rule
22.	$t\mu\exists\nu \Rightarrow \exists <^{\mu}_{\nu}$	a.f. 1.21: WA, $\exists I$
23.	$\exists\mu\exists\nu \Rightarrow \exists <^{\mu}_{\nu}$	a.f. 1.22: WA, $\exists I$
24.	$\forall <^{\mu}_{\nu} \Rightarrow t\mu, s\nu$	$\overline{< E1}$
25.	$\forall <^{\mu}_{\nu} \Rightarrow \forall\mu, t\nu$	a.f. 1.24: SC, $\forall E$
26.	$\forall <^{\mu}_{\nu} \Rightarrow \forall\mu, \forall\nu$	a.f. 1.25: SC, $\forall E$
27.	$\forall <^{\mu}_{\nu} \Rightarrow \exists\mu\forall\nu$	$\overline{< E2}$
28.	$\forall <^{\mu}_{\nu} \Rightarrow \forall\mu\exists\nu$	$< E2$
29.	$\forall <^{\mu}_{\nu} \Rightarrow t\mu, \exists\nu$	$< E2, \forall E$
30.	$\forall <^{\mu}_{\nu} \Rightarrow \exists\mu, \exists\nu$	$< E2, \forall E, \exists I$
31.	$t <^{\mu}_{\nu} \Rightarrow t\mu, \exists\nu$	$\exists I$
32.	$t <^{\mu}_{\nu} \Rightarrow \exists\mu, \exists\nu$	$\exists I, \exists I$
33.	$t <^{\mu}_{\nu} \Rightarrow *$	1.7,8
34.	$\exists <^{\mu}_{\nu} \Rightarrow \exists\mu, \exists\nu$	$< E1$
35.	$\exists <^{\mu}_{\nu} \Rightarrow *$	1.10,11

Beispiel 1

$$t_1, \exists < {}^{8}_{9}\forall_4, \forall_5 \exists < {}^{6}_{7}\forall < {}^{2}_{3} - a - F_{123456789}$$
$$\vdash$$
$$t_3, \forall < {}^{4}_{5}\exists < {}^{1}_{2}, \exists_6, \exists_7, \exists_8, \exists_9 - a - F_{123456789}$$

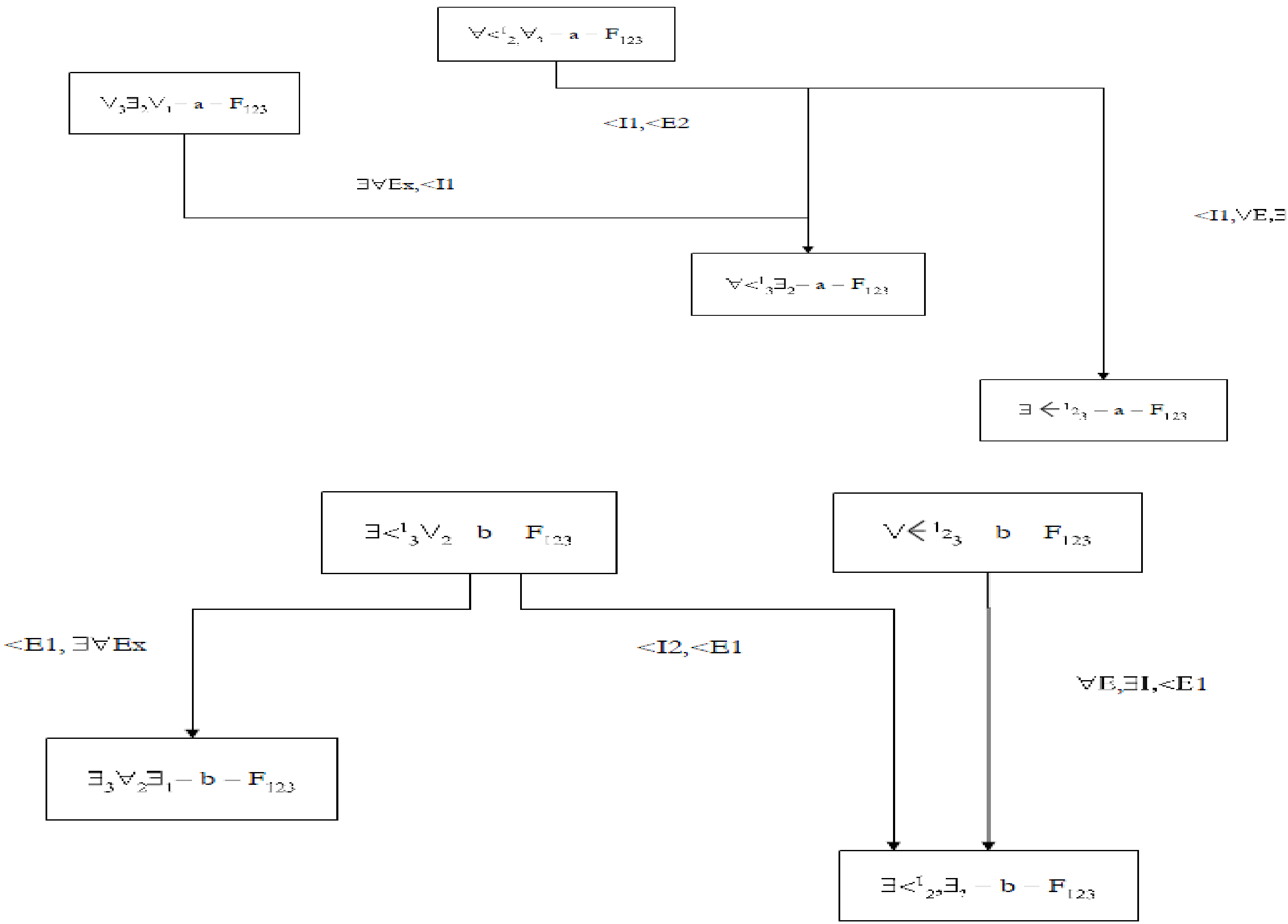
no.	**pole**	**rule**
1.	$t_1, \exists < {}^{8}_{9}\forall_4, \forall_5 \exists < {}^{6}_{7}\forall < {}^{2}_{3} - a - F_{123456789}$	A
2.	$t_1, \exists < {}^{8}_{9}\forall < {}^{4}_{5}\exists < {}^{6}_{7}\forall < {}^{2}_{3} - a - F_{123456789}$	$< I1$
3.	$t <^{1}_{\substack{2\\3}}, \exists < {}^{8}_{9}\forall < {}^{4}_{5}\exists < {}^{6}_{7} - a - F_{123456789}$	$\forall E$
4.	$t <^{1}_{\substack{2\\3}}, \exists < {}^{8}_{9}\forall < {}^{4}_{5}\exists_6, \exists_7 - a - F_{123456789}$	$< E1$
5.	$t <^{1}_{\substack{2\\3}}, \exists_8, \exists_9 \forall < {}^{4}_{5}\exists_6, \exists_7 - a - F_{123456789}$	$< E1$
6.	$t_3, \exists < {}^{1}_{2}, \exists_8, \exists_9 \forall < {}^{4}_{5}\exists_6, \exists_7 - a - F_{123456789}$	$\exists I$
7.	$t_3, \forall < {}^{4}_{5}\exists < {}^{1}_{2}, \exists_6, \exists_7, \exists_8, \exists_9 - a - F_{123456789}$	$\exists\forall Ex$

Beispiel 2

$$\exists_4\forall_5\exists_3\forall < \tfrac{1}{2} - b - F_{12345} \quad \not\vdash \quad \forall_1\exists_4\forall_5\exists < \tfrac{2}{3} - b - F_{12345}$$

no.	**pole**	**rule**
1.	$\exists_4\forall_5\exists_3\forall < \frac{1}{2} - b - F_{12345}$	A
2.	$\exists_4\forall_5\exists \leftarrow\!\!< \genfrac{}{}{0pt}{}{1}{\frac{2}{3}} - b - F_{12345}$	$< E1$
3.	$t_1, \exists_4\forall_5\exists < \frac{2}{3} - b - F_{12345}$	$\overline{\exists E}$

Polbeziehungen



Konträre Pole

$$
\begin{array}{lcl}
\forall_3\exists_2\forall_1 - a - F_{123} & \lhd\!-\!\rhd & \exists_3\forall_2\exists_1 - b - F_{123} \\
\forall < \frac{1}{3}\exists_2 - a - F_{123} & \lhd\!-\!\rhd & \exists < \frac{1}{3}\forall_2 - b - F_{123} \\
\forall < \frac{1}{2},\forall_3 - a - F_{123} & \lhd\!-\!\rhd & \exists < \frac{1}{2},\exists_3 - b - F_{123} \\
\exists < \frac{1}{\frac{2}{3}} - a - F_{123} & \lhd\!-\!\rhd & \forall < \frac{1}{\frac{2}{3}} - b - F_{123} \\
\forall_3\exists_2\forall_1 - a - F_{123} & \lhd\!-\!\rhd & \exists < \frac{1}{3}\forall_2 - b - F_{123} \\
\forall < \frac{1}{2},\forall_3 - a - F_{123} & \lhd\!-\!\rhd & \exists < \frac{1}{3}\forall_2 - b - F_{123} \\
\forall < \frac{1}{2},\forall_3 - a - F_{123} & \lhd\!-\!\rhd & \forall < \frac{1}{\frac{2}{3}} - b - F_{123}.
\end{array}
$$

CL, S. 53 (LW an Russell 1913):

„And this is the one symbolic rule: write the prop[osition] down in the ab-notation, trace all Connections (of Poles) from the outside to the inside Poles: Then if the b-Pole is connected to such *groups of inside Poles ONLY as contain opposite poles* […], then the whole prop[osition]is a true, logical prop[osition]. […]

Of course the rule I have given applies first of all only for what you called elementary prop[osition]. But it is easy to see that it must also apply to all others.“

MIN-Polgruppen

$$
\begin{array}{ll}
1. & a - \{\forall_3\exists_2\forall_1 - a - F_{123},\quad \forall < \tfrac{1}{3}\exists_2 - a - F_{123},\quad \forall < \tfrac{1}{2},\forall_3 - a - F_{123},\quad \exists <^{1}_{\substack{2\\3}} - a - F_{123}\}, \\
3. & a - \{\forall_3\exists_2\forall_1 - a - F_{123},\quad \forall < \tfrac{1}{3}\exists_2 - a - F_{123},\quad \exists < \tfrac{1}{2},\exists_3 - b - F_{123},\quad \exists <^{1}_{\substack{2\\3}} - a - F_{123}\}, \\
5. & a - \{\exists_3\forall_2\exists_1 - b - F_{123},\quad \forall < \tfrac{1}{3}\exists_2 - a - F_{123},\quad \forall < \tfrac{1}{2},\forall_3 - a - F_{123},\quad \exists <^{1}_{\substack{2\\3}} - a - F_{123}\}, \\
10. & a - \{\exists_3\forall_2\exists_1 - b - F_{123},\quad \forall < \tfrac{1}{3}\exists_2 - a - F_{123},\quad \exists < \tfrac{1}{2},\exists_3 - b - F_{123},\quad \exists <^{1}_{\substack{2\\3}} - a - F_{123}\}, \\
15. & a - \{\exists_3\forall_2\exists_1 - b - F_{123},\quad \exists < \tfrac{1}{3}\forall_2 - b - F_{123},\quad \exists < \tfrac{1}{2},\exists_3 - b - F_{123},\quad \exists <^{1}_{\substack{2\\3}} - a - F_{123}\},
\end{array}
$$

$$
\begin{array}{ll}
6. & b - \{\forall_3\exists_2\forall_1 - a - F_{123},\quad \forall < \tfrac{1}{3}\exists_2 - a - F_{123},\quad \exists < \tfrac{1}{2},\exists_3 - b - F_{123},\quad \forall <^{1}_{\substack{2\\3}} - b - F_{123}\}, \\
13. & b - \{\exists_3\forall_2\exists_1 - b - F_{123},\quad \forall < \tfrac{1}{3}\exists_2 - a - F_{123},\quad \exists < \tfrac{1}{2},\exists_3 - b - F_{123},\quad \forall <^{1}_{\substack{2\\3}} - b - F_{123}\}, \\
16. & b - \{\exists_3\forall_2\exists_1 - b - F_{123},\quad \exists < \tfrac{1}{3}\forall_2 - b - F_{123},\quad \exists < \tfrac{1}{2},\exists_3 - b - F_{123},\quad \forall <^{1}_{\substack{2\\3}} - b - F_{123}\}.
\end{array}
$$

Subkonträre Pole

$$\forall_3\exists_2\forall_1 - a - F_{123} \quad \diamond - \diamond \quad \exists_3\forall_2\exists_1 - b - F_{123}$$

$$\forall < {}^{1}_{3}\exists_2 - a - F_{123} \quad \diamond - \diamond \quad \exists < {}^{1}_{3}\forall_2 - b - F_{123}$$

$$\forall < {}^{1}_{2},\forall_3 - a - F_{123} \quad \diamond - \diamond \quad \exists < {}^{1}_{2},\exists_3 - b - F_{123}$$

$$\exists <_{\substack{1\\2\\3}} - a - F_{123} \quad \diamond - \diamond \quad \forall <_{\substack{1\\2\\3}} - b - F_{123}$$

$$\forall < {}^{1}_{3}\exists_2 - a - F_{123} \quad \diamond - \diamond \quad \exists_3\forall_2\exists_1 - b - F_{123}$$

$$\forall < {}^{1}_{3}\exists_2 - a - F_{123} \quad \diamond - \diamond \quad \exists < {}^{1}_{2},\exists_3 - b - F_{123}$$

$$\exists <_{\substack{1\\2\\3}} - a - F_{123} \quad \diamond - \diamond \quad \exists < {}^{1}_{2},\exists_3 - b - F_{123}$$

MIN-PG $\Rightarrow$ PRIM-PG

TABLE 1

1. $a - \{\forall_3\exists_2\forall_1 - a - F_{123},\ \ \forall < \tfrac{1}{3}\exists_2 - a - F_{123},\ \forall < \tfrac{1}{2},\forall_3 - a - F_{123}, \exists <^{\frac{1}{2}}_{3} - a - F_{123}\}$,
2. $a - \{\forall_3\exists_2\forall_1 - a - F_{123},\ \ \forall < \tfrac{1}{3}\exists_2 - a - F_{123},\ \exists < \tfrac{1}{2},\exists_3 - b - F_{123}, \exists <^{\frac{1}{2}}_{3} - a - F_{123}\}$,
3. $a - \{\exists_3\forall_2\exists_1 - b - F_{123},\ \ \forall < \tfrac{1}{3}\exists_2 - a - F_{123},\ \forall < \tfrac{1}{2},\forall_3 - a - F_{123}, \exists <^{\frac{1}{2}}_{3} - a - F_{123}\}$,
4. $a - \{\exists_3\forall_2\exists_1 - b - F_{123},\ \ \forall < \tfrac{1}{3}\exists_2 - a - F_{123},\ \exists < \tfrac{1}{2},\exists_3 - b - F_{123}, \exists <^{\frac{1}{2}}_{3} - a - F_{123}\}$,

5. $a - \{\exists_3\forall_2\exists_1 - b - F_{123},\ \ \exists < \tfrac{1}{3}\forall_2 - b - F_{123},\ \exists < \tfrac{1}{2},\exists_3 - b - F_{123}, \exists <^{\frac{1}{2}}_{3} - a - F_{123}\}$.

TABLE 2

6. MR3: 1/2 $a - \{\forall_3\exists_2\forall_1 - a - F_{123},\ \ \forall < \tfrac{1}{3}\exists_2 - a - F_{123},\ \exists <^{\frac{1}{2}}_{3} - a - F_{123}\}$,
7. MR3.: 1/3 $a - \{\forall < \tfrac{1}{3}\exists_2 - a - F_{123}, \forall < \tfrac{1}{2},\forall_3 - a - F_{123}, \exists <^{\frac{1}{2}}_{3} - a - F_{123}\}$,
8. MR3: 2/4 $a - \{\forall < \tfrac{1}{3}\exists_2 - a - F_{123}, \exists < \tfrac{1}{2},\exists_3 - b - F_{123}, \exists <^{\frac{1}{2}}_{3} - a - F_{123}\}$,
9. MR3: 3/4 $a - \{\exists_3\forall_2\exists_1 - b - F_{123},\ \ \forall < \tfrac{1}{3}\exists_2 - a - F_{123},\ \exists <^{\frac{1}{2}}_{3} - a - F_{123}\}$,
10. MR3: 4/5 $a - \{\exists_3\forall_2\exists_1 - b - F_{123},\ \ \exists < \tfrac{1}{2},\exists_3 - b - F_{123}, \exists <^{\frac{1}{2}}_{3} - a - F_{123}\}$.

TABLE 3

11. MR3: 6/9 $a - \{\forall < \tfrac{1}{3}\exists_2 - a - F_{123}, \exists <^{\frac{1}{2}}_{3} - a - F_{123}\}$,
12. MR3 8/10 $a - \{\exists < \tfrac{1}{2},\exists_3 - b - F_{123}, \exists <^{\frac{1}{2}}_{3} - a - F_{123}\}$.

TABLE 4

13. MR3: 11/12 $a - \{\exists <^{\frac{1}{2}}_{3} - a - F_{123}\}$.

b-pole-groups:

TABLE 1

1. $b - \{\forall_3\exists_2\forall_1 - a - F_{123},\ \ \forall < \tfrac{1}{3}\exists_2 - a - F_{123},\ \exists < \tfrac{1}{2},\exists_3 - b - F_{123}, \forall <^{\frac{1}{2}}_{3} - b - F_{123}\}$,
2. $b - \{\exists_3\forall_2\exists_1 - b - F_{123},\ \ \forall < \tfrac{1}{3}\exists_2 - a - F_{123},\ \exists < \tfrac{1}{2},\exists_3 - b - F_{123}, \forall <^{\frac{1}{2}}_{3} - b - F_{123}\}$,
3. $b - \{\exists_3\forall_2\exists_1 - b - F_{123},\ \ \exists < \tfrac{1}{3}\forall_2 - b - F_{123},\ \exists < \tfrac{1}{2},\exists_3 - b - F_{123}, \forall <^{\frac{1}{2}}_{3} - b - F_{123}\}$.

TABLE 2

4. MR3: 1/2 $b - \{\forall < \tfrac{1}{3}\exists_2 - a - F_{123}, \exists < \tfrac{1}{2},\exists_3 - b - F_{123}, \forall <^{\frac{1}{2}}_{3} - b - F_{123}\}$,
5. MR3: 2/3 $b - \{\exists_3\forall_2\exists_1 - b - F_{123},\ \ \exists < \tfrac{1}{2},\exists_3 - b - F_{123}, \forall <^{\frac{1}{2}}_{3} - b - F_{123}\}$.

TABLE 3

6. MR3.: 4/5 $b - \{\exists < \tfrac{1}{2},\exists_3 - b - F_{123}, \forall <^{\frac{1}{2}}_{3} - b - F_{123}\}$.

TABLE 4

7. MR1: 6 $b - \{\forall <^{\frac{1}{2}}_{3} - b - F_{123}\}$.

ab-Symbol, Paraphrase

ab-Symbol:

$$a - \{\exists <^{1}_{\frac{2}{3}} - a - F_{123}\},$$
$$b - \{\forall <^{1}_{\frac{2}{3}} - b - F_{123}\}.$$

Paraphrase: Eine Instanz von

$$\forall z \exists y \forall x Fxyz \wedge \forall x \exists y Fxyx \wedge \forall x \forall y Fxxy \vee \exists x Fxxx$$

- ist wahr gdw. für mindestens ein Gegenstand, derselbe an der 1., 2., 3. Argumentstelle von F_{123}, F_{123} wahr ist.
- falsch gdw. für alle Gegenstände, dieselben an der 1., 2., 3. Argumentstelle von F_{123}, F_{123} falsch ist.

Molekulare Wirkungsbereiche

$\exists x(Fx \wedge Gx)$

ab-Symbol:

$$\text{a-}\{\ \exists < \begin{array}{l} 1\ -a-F_1 \\ 1\ -a-G_1 \end{array} \},\quad \text{b-}\{\ \forall \begin{array}{l} / \\ \backslash \end{array} \begin{array}{l} 1\ -b-F_1 \\ 1\ -b-G_1 \end{array} \}.$$

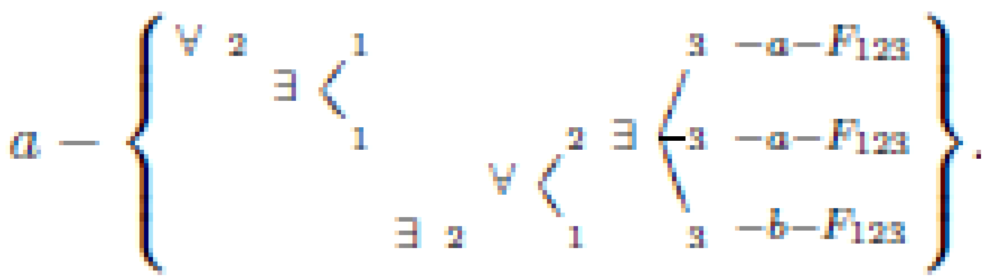
Paraphrase: Eine Instanz von $\exists x(Fx \wedge Gx)$ ist

- wahr gdw. für mindestens ein x, dasselbe an der 1. Argumentstelle der 1-stelligen Aussagefunktion Fx und der 1. Argumentstelle der 1-stelligen Aussagefunktion Gx, Fx wahr und Gx wahr ist.
- falsch gdw. für alle x, aufgeteilt auf die 1. Argumentstelle der 1-stelligen Aussagefunktion Fx und der 1. Argumentstelle der 1-stelligen Aussagefunktion Gx, Fx falsch und Gx falsch ist.

$$\forall x_1 \exists y_1 \exists y_2 \forall x_2 \exists y_3 (F y_1 x_1 y_3 \land F y_1 x_2 y_3 \land \neg F x_2 y_2 y_3)$$

Für $I = \{c_1\}$, $I = \{c_1, c_2\}$ kein Modell,

Für $I = \{c_1, c_2, c_3\}$ gibt es $2^{27} = 134\,217\,728$ $\mathfrak{I}$.



Konstruktion von Modellen

$\forall_2$	$\exists <\, \substack{1\\1}$	$\exists_2$	$\forall <\, \substack{2\\1}$	$\exists <\, \substack{3\\3\\3}$
c_1	c_1	c_3	c_1	c_2
c_1	c_1	c_3	c_2	c_2
c_1	c_1	c_3	c_3	c_1
c_2	c_2	c_1	c_1	c_3
c_2	c_2	c_1	c_2	c_1
c_2	c_2	c_1	c_3	c_1
c_3	c_3	c_2	c_1	c_3
c_3	c_3	c_2	c_2	c_2
c_3	c_3	c_2	c_3	c_3

$a - F_{123}$	$a - F_{123}$	$b - F_{123}$
$c_1c_1c_2$	$c_1c_1c_2$	$c_1c_3c_2$
$c_1c_1c_2$	$c_1c_2c_2$	$c_2c_3c_2$
$c_1c_1c_1$	$c_1c_3c_1$	$c_3c_3c_1$
$c_2c_2c_3$	$c_2c_1c_3$	$c_1c_1c_3$
$c_2c_2c_1$	$c_2c_2c_1$	$c_2c_1c_1$
$c_2c_2c_1$	$c_2c_3c_1$	$c_3c_1c_1$
$c_3c_3c_3$	$c_3c_1c_3$	$c_1c_2c_3$
$c_3c_3c_2$	$c_3c_2c_2$	$c_2c_2c_2$
$c_3c_3c_3$	$c_3c_3c_3$	$c_3c_2c_3$

Modell

$$
\begin{aligned}
I = \quad & \{c_1, c_2, c_3\}, \\
\Im^*(F) = \quad & \{(c_1,c_1,c_2), \overline{(c_1,c_3,c_2)}, (c_1,c_2,c_2), \overline{(c_2,c_3,c_2)}, (c_1,c_1,c_1), (c_1,c_3,c_1), \overline{(c_3,c_3,c_1)}, \\
& (c_2,c_2,c_3), (c_2,c_1,c_3), \overline{(c_1,c_1,c_3)}, (c_2,c_2,c_1), \overline{(c_2,c_1,c_1)}, (c_2,c_3,c_1), \overline{(c_3,c_1,c_1)}, \\
& (c_3,c_3,c_3), (c_3,c_1,c_3), \overline{(c_1,c_2,c_3)}, (c_3,c_3,c_2), (c_3,c_2,c_2), \overline{(c_2,c_2,c_2)}, \overline{(c_3,c_2,c_3)}\}.
\end{aligned}
$$

Kalkül: Erweiterung

P E: $\{A,B\} \vdash \{B\}$	PG I: $\{B\} \vdash \{B\}, \{C\}$
P I: $\{A,B\} \vdash \{A,A,B\}$	PG E: $\{B\}, \{B\} \vdash \{B\}$
$\top$ I: $\{B\} \vdash \{B,A\},\{B,\overline{A}\}$	$\perp$ E: $\{B\},\{A,\overline{A}\} \vdash \{B\}$

$\exists\forall Ex$: $\exists\mu \ldots \forall\nu \vdash \forall\nu \ldots \exists\mu$	
$\forall E1$: $\forall\mu A(\mu) \vdash A(\mu/t)$	$\exists I1$: $A(t) \vdash \exists\mu A(t,t/\mu)$
$\forall E2$: $\forall\mu \ldots \forall\nu A(\mu,\nu) \vdash \forall\mu A(\mu, \nu/\mu)$	$\exists I2$: $\exists\mu A(\mu) \vdash \exists\mu \ldots \exists\nu A(\mu,\mu/\nu)$
$\forall E3$: $\exists\mu \ldots \forall\nu A(\mu,\nu) \vdash \exists\mu A(\nu/\mu)$	$\exists I3$: $\forall\mu \vdash \forall\mu \ldots \exists\nu A(\mu, \mu/\nu)$
$\wedge E$: $\bigwedge A, B \vdash A$	$\vee I$: $A \vdash \bigvee A, B$
$\wedge I$: $A \vdash \bigwedge A, A$	$\vee E$: $\bigvee A, A \vdash A$
$\top I$: $A \vdash \bigwedge A, \top$	$\perp E$: $\bigvee A, \perp \vdash A$

Komplexität

Interne Beziehungen nicht mehr reduzible auf die Beziehung zwischen 2 Polen:

1. Mehrere Pole bzw. Polgruppen können einen Pol implizieren;
2. Ein Pol kann mehrere Pole bzw. Polgruppen implizieren;
3. Kontrarität kann nicht allein daran gemessen werden, ob ein Pol ein dem anderen Pol entgegengesetzen Pol impliziert;
4. Entscheidung über Kontrarität erfordert u.U. Polerweiterung;
5. Anzahl der Polgruppen kann wachsen, z.B. :
   $$\forall \left\langle \begin{array}{l} 1\ -b-F_1 \\ 1\ -b-G_1 \end{array} \right. \quad \vdash \{\ldots, \exists_1\text{-a-}F_1, \ldots\},\{\ldots\ \exists_1\text{-a-}G_1, \ldots\};$$
6. Nicht nur Polgruppen, auch Wirkungsbereiche müssten in eindeutige Repräsentanten umgeformt und dafür erweitert werden.

$$\exists x \exists y Fxy$$

-||-

$$\exists x(Fxx \wedge \exists y \neg Fxy \wedge \exists y \neg Fyx) \vee$$
$$\exists x(\neg Fxx \wedge \exists y Fxy \wedge \exists y Fyx) \vee$$
$$\exists x(\exists y Fxy \wedge \forall y \neg Fyx) \vee$$
$$\exists x(\forall y \neg Fxy \wedge \exists y Fyx) \vee$$
$$\forall x(\exists y Fxy \vee \exists y Fyx)$$

-||-

CFOLDNF mit $2^8 = 256$ Disjunkten,

davon 68 nicht-kontradiktorisch

Nicht-elementare Pole

$cs1\colon \exists x(\forall y Fxy \wedge \forall y Fyx)$

$cs2\colon \exists x(\forall y Fxy \wedge \exists y \neg Fyx)$

$cs3\colon \exists x(\exists y \neg Fxy \wedge \forall y Fyx)$

$cs4\colon \exists x(Fxx \wedge \exists y \neg Fxy \wedge \exists y \neg Fyx)$

$cs5\colon \exists x(\neg Fxx \wedge \exists y Fxy \wedge \exists y Fyx)$

$cs6\colon \exists x(\exists y Fxy \wedge \forall y \neg Fyx)$

$cs7\colon \exists x(\forall y \neg Fxy \wedge \exists y Fyx)$

$cs8\colon \exists x(\forall y \neg Fxy \wedge \forall y \neg Fyx)$

$\neg cs1\colon \forall x(\exists y \neg Fxy \vee \exists y \neg Fyx)$

$\neg cs2\colon \forall x(\exists y \neg Fxy \vee \forall y Fyx)$

$\neg cs3\colon \forall x(\forall y Fxy \vee \exists y \neg Fyx)$

$\neg cs4\colon \forall x(\neg Fxx \vee \forall y Fxy \vee \forall y Fyx)$

$\neg cs5\colon \forall x(Fxx \vee \forall y \neg Fxy \vee \forall y \neg Fyx)$

$\neg cs6\colon \forall x(\forall y \neg Fxy \vee \exists y Fyx)$

$\neg cs7\colon \forall x(\exists y Fxy \vee \forall y \neg Fyx)$

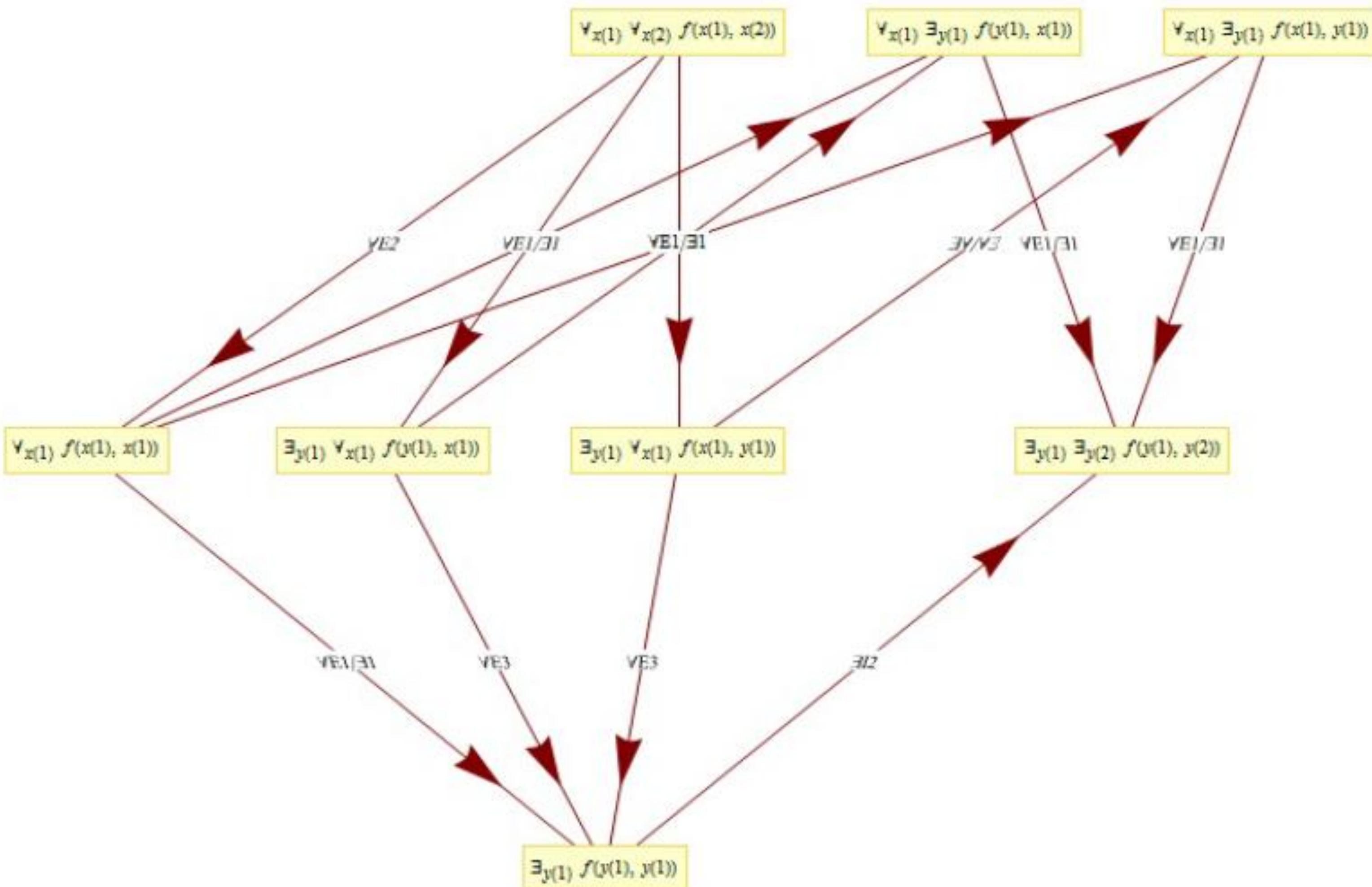
$\neg cs8\colon \forall x(\exists y Fxy \vee \exists y Fyx)$

$$\exists x Fx \wedge \exists x Gx$$

-||-

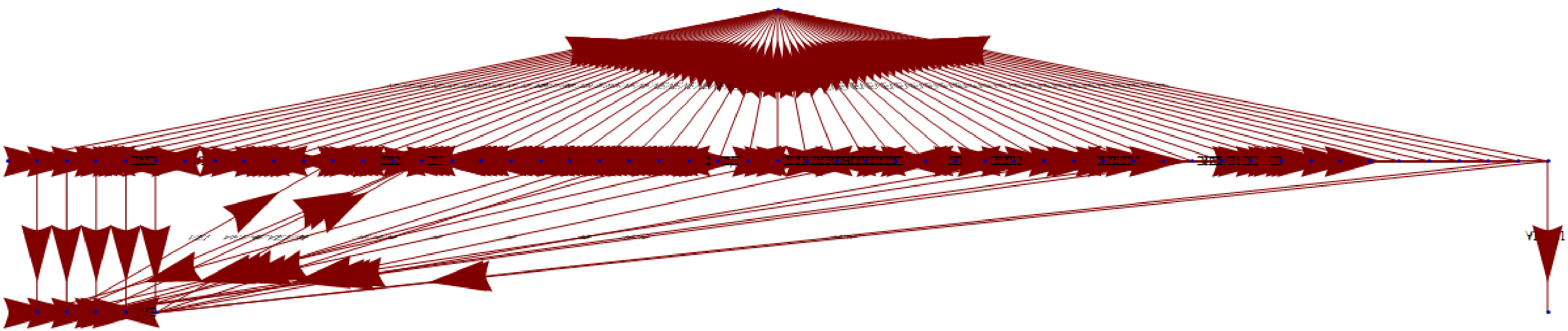
$$\exists x(Fx \wedge Gx) \wedge \exists x(Fx \wedge \neg Gx) \wedge \exists x(\neg Fx \wedge Gx) \wedge \exists x(\neg Fx \wedge \neg Gx) \vee$$
$$\exists x(Fx \wedge Gx) \wedge \exists x(Fx \wedge \neg Gx) \wedge \exists x(\neg Fx \wedge Gx) \wedge \forall x(Fx \vee Gx) \vee$$
$$\exists x(Fx \wedge Gx) \wedge \exists x(Fx \wedge \neg Gx) \wedge \forall x(Fx \vee \neg Gx) \wedge \exists x(\neg Fx \wedge \neg Gx) \vee$$
$$\exists x(Fx \wedge Gx) \wedge \exists x(Fx \wedge \neg Gx) \wedge \forall x(Fx \vee \neg Gx) \wedge \forall x(Fx \vee Gx) \vee$$
$$\exists x(Fx \wedge Gx) \wedge \forall x(\neg Fx \vee Gx) \wedge \exists x(\neg Fx \wedge Gx) \wedge \exists x(\neg Fx \wedge \neg Gx) \vee$$
$$\exists x(Fx \wedge Gx) \wedge \forall x(\neg Fx \vee Gx) \wedge \exists x(\neg Fx \wedge Gx) \wedge \forall x(Fx \vee Gx) \vee$$
$$\exists x(Fx \wedge Gx) \wedge \forall x(\neg Fx \vee Gx) \wedge \forall x(Fx \vee \neg Gx) \wedge \exists x(\neg Fx \wedge \neg Gx) \vee$$
$$\exists x(Fx \wedge Gx) \wedge \forall x(\neg Fx \vee Gx) \wedge \forall x(Fx \vee \neg Gx) \wedge \forall x(Fx \vee Gx) \vee$$
$$\forall x(\neg Fx \vee \neg Gx) \wedge \exists x(Fx \wedge \neg Gx) \wedge \exists x(\neg Fx \wedge Gx) \wedge \exists x(\neg Fx \wedge \neg Gx) \vee$$
$$\forall x(\neg Fx \vee \neg Gx) \wedge \exists x(Fx \wedge \neg Gx) \wedge \exists x(\neg Fx \wedge Gx) \wedge \forall x(Fx \vee Gx)$$

Beweisbaum: $\forall_{x(1)} \forall_{x(2)} f(x(1), x(2))$



Beweisbaum:

$$\exists_{y(2)} \left(\exists_{y(1)} f(y(2), y(1)) \bigwedge \forall_{x(1)} \left(\exists_{y(3)} f(y(3), x(1)) \bigvee f(y(2), x(1))\right)\right)$$



224 Einträge in 8 sec.

Ausblick: FOL-Decider

1. Nicht Lösung des Äquivalenzproblems, sondern des Entscheidungsproblems für Widersprüchlichkeit.
2. Nicht mehr zwingend logisch-äquivalente Umformung, sondern sat-äquivalente Umformung.
3. Keine ab-Notation, sondern FOL-interne sat-Äquivalenzumformung.
4. $\phi \Rightarrow$ optimierte FOLDNF: Entscheidung der Widersprüchlichkeit der einzelnen Disjunkte D durch endlichen Beweisbaum mit einem Minimum an Komplexität.

FOL-Model Check und FOL-Optimizer

- FOL-Model Check: Konstruktion eines Modelles für erfüllbare Formeln auf Basis von FOLDNF.
- FOL-Optimizer: Realisierung eines Analogon des Quine-McCluskey Algorithmus für FOL.
- Eindeutigkeitsforderung ist aus praktischen Gründen (wg. Unumgänglichkeit der Komplexitätserweiterung) durch Minimalitätsforderung zu ersetzen.
- Der FOL-Optimizer ist ein Algorithmus zur Erzeugung minimaler FOLDNF.